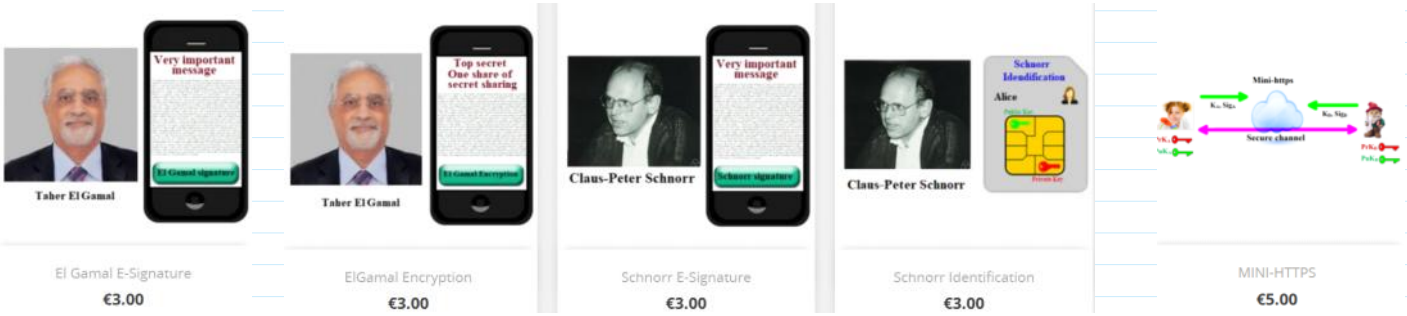
During the exam you must solve the following 5 problems.



And answer to 2 questions from the declared list.

DEF-Homomophism. ElG-Enc

Compatibility relations of modular arithmetic:
(a + b)mod p = (a mod p + b mod p) mod p.
(a * b) mod p =((a mod p) * (b mod p)) mod p.
$a^p \bmod p = (a \bmod p)^p \bmod p$.

Fermat little theorem: If p is prime, then for any integer a holds $a^p = a \bmod p$.

1. We may assume that a is in the range $0 \le a \le p - 1$.
This is a simple consequence of the laws of modular arithmetic; we are simply saying that we may first reduce a modulo p since
$a^p \bmod p = ((a) \bmod p)^p \bmod p$.
1. It suffices to prove that for a in the range $1 \le a \le p$

$a^p = a \bmod p \quad / \bar{a}^1 \bmod p$

$a^p \cdot \bar{a}^1 = a \cdot \bar{a}^1 \bmod p$

$0 < a < p. \quad a^{p-1} = 1 \bmod p$ $\quad a^0 = a^{p-1} = 1 \bmod p.$

Indeed, if the previous assertion holds for such a, multiplying both sides by a yields the original form of the theorem.

Computation of exponents mod (p-1):

$s = xh + r \longrightarrow g^{s \bmod (p-1)} \bmod p$

$g^{p-1} = 1 \bmod p \;\&\; g^0 = 1 \bmod p \Rightarrow 0 \equiv p-1$

$a^z \bmod p = a^{z \bmod (p-1)} \bmod p$.

```
>> p=genstrongprime(28)   // strong prime number generation
p = 215393099             // p - is strong prime iff p=2*q+1, when q is also prime: q=(p-1)/2
>> q=(p-1)/2
q = 107696549
                          // finding a generator g: g - is a generator in Zp* = {1, 2, 3, …, p-1} *mod p
                          // iff g^q =/= 1 & g^2 =/= 1
>> g=2
g = 2
>> mod_exp(g,q,p)
ans = 215393098
>> mod_exp(g,2,p)
ans = 4                   // g=2 is a generator
```

$s = (xh + r) \bmod (p-1)$; $g^s \bmod p = g^{s \bmod (p-1)} \bmod p$.

```
>> x=int64(randi(p-1))
x = 169506978
>> a=mod_exp(g,x,p)
a = 99428799
>> con=concat('hello bob',a)
con = hello bob99428799
>> h=hd28(con)
h = 252819993
>> r=int64(randi(p-1))
r = 96513791
```

```
>> s=int64(x*h+r)
s = 42854753087924945
>> smodpm1=mod(s,p-1)
smodpm1 = 150400265

>> g_s=mod_exp(g,s,p)
g_s = 162111969
>> g_smodpm1=mod_exp(g,smodpm1,p)
g_smodpm1 = 162111969
```

DEF homomorphism

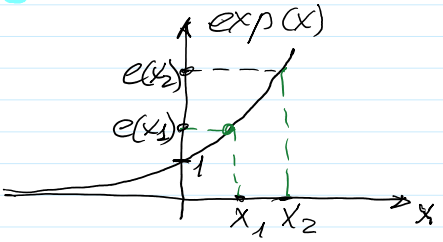
$\exp(x) = e^x$; $\exp: R \to R$; $e = 2,71\ldots$

$x \in R \to e^x \in R$.

$\exp(x_1 + x_2) = e^{(x_1 + x_2)} = e^{x_1} \cdot e^{x_2} = \exp(x_1) \cdot \exp(x_2)$

Additively - multiplicative homomorphism.

Since it is 1-to-1, then it is isomorphism.



$DEF(x) = g^x \bmod p$; p - (strong) prime

g - generator in $\mathbb{Z}_p^* = \{1, 2, 3, \ldots, p-1\}$

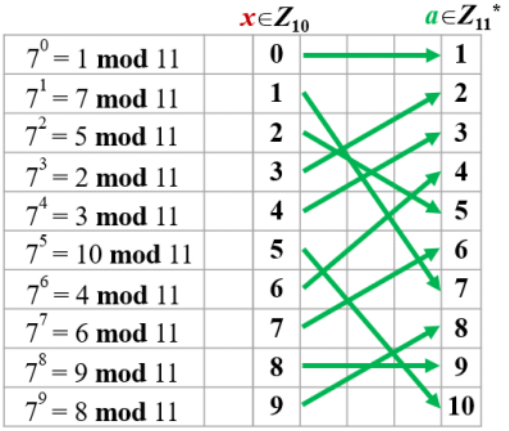
$x \in \mathcal{Z}_{p-1} = \{0, 1, 2, 3, \ldots, p-2\}$; $+ \bmod (p-1)$, $* \bmod (p-1)$, $- \bmod (p-1)$ / $\bmod (p-1)$.

$|\mathcal{Z}_{p-1}| = p-1$

$DEF(x) = a \in \mathcal{Z}_p^* = \{1, 2, 3, \ldots, p-1\}$; $* \bmod p$, $/ \bmod p$.

$|\mathcal{Z}_p^*| = p-1 = |\mathcal{Z}_{p-1}|$

DEF realizes the following mapping DEF: $\mathbf{Z_{p-1}}$ --> $\mathbf{Z_p^*}$
DEF is defined by Public Parameters: **PP** = (***p***, ***g***)
$DEF_{g,p}(x) = g^x \bmod p = a$.



$DEF(x_1 + x_2) = g^{(x_1+x_2) \bmod (p-1)} \bmod p = g^{x_1} \cdot g^{x_2} \bmod p =$

(a * b) mod p =((a mod p) * (b mod p)) mod p.

$= ((g^{x_1} \bmod p) \cdot (g^{x_2} \bmod p)) \bmod p = DEF(x_1) \cdot DEF(x_2)$

Additively – multiplicative homomorphism.
Since it is 1-to-1, then it is isomorphism.

ElGamal Encryption-Decryption

Public Parameters generation **PP** = (***p***, ***g***).

Asymmetric Signing - Verification

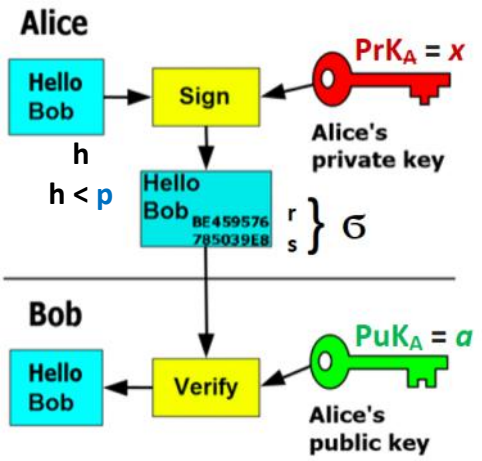
Sign(PrK_A, h) = σ = (r, s)

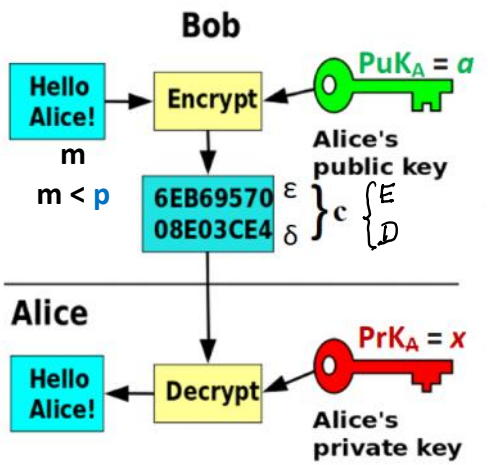
V=Ver(PuK_A, h, σ), V∈{True, False} ≡ {1, 0}

Asymmetric Encryption - Decryption

c=Enc(PuK_A, m)

m=Dec(PrK_A, c)





ElGamal Cryptosystem

1.Public Parameters generation $PP = (p, g)$.

Generate strong prime number **p**: >> p=genstrongprime(28) % strong prime of 28 bit length

Find a generator **g** in $Z_p^* = \{1, 2, 3, \ldots, p-1\}$ using condition.

Strong prime $p=2q+1$, where q is prime, then g is a generator of Z_p^* iff

$g^q \neq 1 \bmod p$ and $g^2 \neq 1 \bmod p$.

Declare **Public Parameters** to the network $PP = (p, g)$; **p= 268435019; g=2;**

2^28-1= 268,435,455

```
>> 2^28-1
ans = 2.6844e+08
>> int64(2^28-1)
ans = 268435455
```

PrK = *x* <-- randi(Z_p^*) ==> PuK = *a* = $g^x \bmod p$

Asymmetric Encryption-Decryption: El-Gamal Encryption-Decryption

p=268435019; g=2;

Let message ***m*~** needs to be encrypted, then it must be encoded in decimal number ***m***: 1< ***m*** < ***p***.
E.g. ***m*** = 111222. Then ***m*** **mod** ***p*** = ***m***.

$\mathcal{I}_p^* = \{1, 2, 3, \ldots, p-1\}$; * mod p

A : $PuK_A = a$ → B : is able to encrypt m to A: $m < p$

B: $i \leftarrow randi(\mathcal{I}_p^*)$

$E = m \cdot a^i \bmod p$

$D = g^i \bmod p$

$c = (E, D)$ →

A : is able to decrypt $c = (E, D)$ using her $PrK_A = x$.

1. $D^{-x \bmod (p-1)} \bmod p$
2. $E \cdot D^{-x} \bmod p = m$

$(-x) \bmod (p-1) = (0 - x) \bmod (p-1) = (p-1-x) \bmod (p-1)$

```
> x=123
x = 123
>> pp=127
pp = 127
```

$(-x) \bmod (p-1) = (0 - x) \bmod (p-1) =$

$= (p-1-x) \bmod (p-1)$

$(p-1) \bmod (p-1) = 0$ since $\frac{p-1}{p-1} = 1$, remainder 0

$(-x) \bmod (p-1) = (p-1-x)$

$D^{-x} \bmod (p-1) = D^{p-1-x} \bmod (p-1)$

$>> D\_mx = mod\_exp(D, p-1-x, p)$

2. $E \cdot D^{-x} \bmod p = m$

```
x = 123
>> pp=127
pp = 127
>> isprime(pp)
ans = 1
>> mx=mod(-x,pp-1)
mx = 3
>> mod(x+mx,pp-1)
ans = 0
```

D^{-x} mod ***p*** computation using Fermat theorem:
If ***p*** is prime, then for any integer ***a*** in Z_p^* holds **$a^{p-1} = 1 \bmod p$**.

$D^{p-1} = 1 \bmod p \quad / \cdot D^{-x \bmod (p-1)} \bmod p$

$D^{p-1} \cdot D^{-x} = 1 \cdot D^{-x} \bmod p \Rightarrow D^{p-1-x} = D^{-x} \bmod p$

$$\boxed{D^{-x} \bmod p = D^{p-1-x} \bmod p}$$

Correctness

$Enc(PuK_A = a, i, m) = c = (E, D) = (E = m \cdot a^i \bmod p; D = g^i \bmod p)$

$Dec(PrK_A = x, c) = E \cdot D^{-x} \bmod p = m \cdot a^i \cdot (g^i)^{-x} \bmod p =$

$= m \cdot (\underbrace{g^x}_{a})^i \cdot g^{-ix} = m \cdot g^{xi} \cdot g^{-ix} = m \cdot g^{xi-ix} \bmod p = m \cdot g^0 \bmod p =$

$= m \cdot 1 \bmod p = m \bmod p = m = 111222$

since $m < p$

$27 \,|\, 5$: $27 - 25 = 2$, quotient 5

If $m > p \rightarrow m \bmod p \neq m$; $27 \bmod 5 = 2 \neq 27$.
If $m < p \rightarrow m \bmod p = m$; $19 \bmod 31 = 19$.
Decryption is correct if $m < p$.

ASCII: 8 bits per char.
$\frac{2048}{8} = 256$ char.

Homomorphic Encryption

Let m_1 and m_2 have to be encrypted.

$Enc(PuK_A = a, i_1, m_1) = c_1 = (E_1, D_1) = (E_1 = m_1 a^{i_1} \bmod p, D_1 = g^{i_1} \bmod p)$

$Enc(PuK_A = a, i_2, m_2) = c_2 = (E_2, D_2) = (E_2 = m_2 a^{i_2} \bmod p, D_2 = g^{i_2} \bmod p)$

$$c_{12} = c_1 \circ c_2 = (E_1, D_1) \cdot (E_2, D_2) = (E_1 \circ E_2, D_1 \circ D_2) = (E_{12}, D_{12}) \pmod p$$

$$E_{12} = m_1 \cdot m_2 \cdot a^{i_1} a^{i_2} \bmod p = m_1 \cdot m_2 \, a^{(i_1+i_2) \bmod (p-1)} \bmod p =$$

$$= m_1 \cdot m_2 \, a^{i_{12}} \bmod p = m_{12} \, a^{i_{12}} \bmod p$$

$$D_{12} = D_1 \circ D_2 = g^{i_1} \cdot g^{i_2} \bmod p = g^{i_1 + i_2} \bmod p = g^{i_{12}} \bmod p$$

$$\begin{cases} m_{12} = m_1 \circ m_2 \bmod p \\ i_{12} = (i_1 + i_2) \bmod (p-1) \end{cases}$$

$$Enc(Puk_A = a, i_{12}, m_1 \cdot m_2) = c_{12} = c_1 \circ c_2 = (E_1 \circ E_2, D_1 \circ D_2) = (E_{12}, D_{12})$$

Multiplicatively homomorphic encryption.

We need an additively-multiplicative encryption.

$$n_1 = g^{m_1} \bmod p \Rightarrow Enc(Puk_A = a, i_1, n_1) = (E_1, D_1) =$$

$$= (E_1 = n_1 \cdot a^{i_1} \bmod p, \; D_1 = g^{i_1} \bmod p).$$

$$n_2 = g^{m_2} \bmod p \Rightarrow Enc(Puk_A = a, i_2, n_2) = (E_2, D_2) =$$

$$= (E_2 = n_2 \cdot a^{i_2} \bmod p, D_2 = g^{i_2} \bmod p.$$

By transforming $\mathbf{m_1}$, $\mathbf{m_2}$ to $\mathbf{n_1}$, $\mathbf{n_2}$ we obtain additively homomorphic encryption with respect to $\mathbf{m_1}$, $\mathbf{m_2}$, by encryption $\mathbf{n_1}$, $\mathbf{n_2}$ in a standard way.

Since

$\mathbf{n_1}*\mathbf{n_2} \bmod \mathbf{p} = \mathbf{g}^{m1}*\mathbf{g}^{m2} \bmod \mathbf{p} = \mathbf{g}^{m1+m2 \bmod (p-1)} \bmod \mathbf{p}$,

Then

$Enc(\mathbf{a}, i_1+i_2 \bmod(\mathbf{p}-1), n_1*n_2 \bmod \mathbf{p}) = Enc(\mathbf{a}, i_1+i_2 \bmod(\mathbf{p}-1), \mathbf{g}^{m1+m2 \bmod (p-1)} \bmod \mathbf{p}) = c_{12} = c_1 * c_2 \bmod \mathbf{p}$.